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Sensitivity of Repeated Eigenvalues to Perturbations

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Two methods for calculating the derivatives of a repeated eigenvalue of viscously damped vibrating systems with respect to a parameter are given. The first method implements the subspace spanned by the eigenvectors corresponding to the repeated eigenvalue. The second method is based on an explicit formula that uses the characteristic equation directly, without explicitly employing the eigenvector data. Examples demonstrate the various results.

I. Introduction

THE sensitivity of the eigenvalues of a vibrating system to perturbations is an important topic in the design and analysis of dynamic systems. It also plays an important role in the development of algorithms for solving some inverse problems in vibrations.[1–3] The basic method for evaluating the derivatives of the eigenvalues of the problem $\boldsymbol{Ax} = \lambda \boldsymbol{x}$ by utilizing right and left eigenvectors goes back to the work of Fox and Kapoor[4] and Nelson.[5] It was pointed out, however,[6–9] that the so-called Nelson's method is not applicable for evaluating the derivatives of repeated eigenvalues. The sensitivity of a repeated eigenvalue of conservative vibrating systems has been determined by Ojalvo,[6] who utilized an auxiliary eigenvalue problem associated with the subspace spanned by the eigenvectors corresponding to the repeated eigenvalue. Chen[7] has further developed this method and obtained the sensitivity of the repeated eigenvalue directly, without using the auxiliary eigenvalue problem. Prells and Friswell[9] derived the sensitivity of a repeated eigenvalue from the characteristic eigenvalue equation, without using the eigenvector data.

Our contribution here is twofold. We first extend the method of Ojalvo[6] to include viscously damped vibratory systems, and then we furnish an explicit formula (41) that determines the sensitivity of a repeated eigenvalue of multiplicity $p \geq 2$.

II. Preliminary Results

Free oscillations of linear vibratory systems are governed by a set of second-order differential equations of the form

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{0}, \qquad \boldsymbol{M}, \boldsymbol{C}, \boldsymbol{K} \in \Re^{n\times n}, \qquad \boldsymbol{x}(t) \in \Re^n \tag{1}$$

where the mass matrix $\boldsymbol{M}$ is symmetric positive definite, the damping matrix $\boldsymbol{C}$ and the stiffness matrix $\boldsymbol{K}$ are symmetric positive semidefinite matrices, and the dots indicate derivatives with respect to time t. Generally, the displacement vector $\boldsymbol{x}$ takes the form

$$\boldsymbol{x}(t) = \boldsymbol{v}e^{\lambda t} \tag{2}$$

where $\boldsymbol{v} \in C^n$ is a constant vector and $\lambda \in C$. Substituting Eq. (2) into Eq. (1) leads to the quadratic eigenvalue problem

$$(\lambda^2\boldsymbol{M} + \lambda\boldsymbol{C} + \boldsymbol{K})\boldsymbol{v} = \boldsymbol{0} \tag{3}$$

In some cases, the matrices in Eq. (3) depend on a parameter α. It is a matter of theoretical and practical interest to determine the sensitivity of the eigenvalues to small perturbations in α. To determine the derivative $\mathrm{d}\lambda/\mathrm{d}\alpha$, we multiply Eq. (3) from the left by $\boldsymbol{v}^T$ and obtain

$$Q \mathrel{\hat{=}} \lambda^2\boldsymbol{v}^T\boldsymbol{M}\boldsymbol{v} + \lambda\boldsymbol{v}^T\boldsymbol{C}\boldsymbol{v} + \boldsymbol{v}^T\boldsymbol{K}\boldsymbol{v} = 0 \tag{4}$$

Then, differentiating Eq. (4) gives

$$\begin{aligned} Q' = {} & \boldsymbol{v}^T(2\lambda\lambda'\boldsymbol{M} + \lambda'\boldsymbol{C})\boldsymbol{v} + \boldsymbol{v}'^T(\lambda^2\boldsymbol{M} + \lambda\boldsymbol{C} + \boldsymbol{K})\boldsymbol{v} \\ & + \boldsymbol{v}^T(\lambda^2\boldsymbol{M}' + \lambda\boldsymbol{C}' + \boldsymbol{K}')\boldsymbol{v} + \boldsymbol{v}^T(\lambda^2\boldsymbol{M} + \lambda\boldsymbol{C} + \boldsymbol{K})\boldsymbol{v}' = 0 \end{aligned} \tag{5}$$

where the prime denotes differentiation with respect to α. If $\boldsymbol{v}'$ is bounded, then Eq. (5) reduces to

$$\lambda'\boldsymbol{v}^T(2\lambda\boldsymbol{M} + \boldsymbol{C})\boldsymbol{v} + \boldsymbol{v}^T(\lambda^2\boldsymbol{M}' + \lambda\boldsymbol{C}' + \boldsymbol{K}')\boldsymbol{v} = 0 \tag{6}$$

by virtue of Eq. (3), yielding

$$\lambda' = \frac{-\boldsymbol{v}^T(\lambda^2\boldsymbol{M}' + \lambda\boldsymbol{C}' + \boldsymbol{K}')\boldsymbol{v}}{\boldsymbol{v}^T(2\lambda\boldsymbol{M} + \boldsymbol{C})\boldsymbol{v}} \tag{7}$$

or, alternatively,

$$\lambda' = -\frac{\partial Q}{\partial \alpha}\bigg/\frac{\partial Q}{\partial \lambda} \tag{8}$$

As noted before, Eqs. (7) and (8) are applicable only if the eigenvalue of the unperturbed system is simple, which implies that $\boldsymbol{v}'$ is bounded.

Example 1: The stiffness and damping matrices defining the three-degree-of-freedom mass–spring–damper system, shown in Fig. 1, depend on a parameter α:

$$\boldsymbol{M} = \boldsymbol{I}_3, \qquad \boldsymbol{K}(\alpha) = \begin{bmatrix} 15 & -5 & -5 \\ -5 & 15+\alpha & -5-\alpha \\ -5 & -5-\alpha & 15+\alpha \end{bmatrix}$$

$$\boldsymbol{C}(\alpha) = \begin{bmatrix} 2\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{9}$$

where $\boldsymbol{I}_n$ is the identity matrix of order n. If $\alpha = 0$, then the eigenvalues of the system are

$$\lambda_1 = \sqrt{-5}, \qquad \lambda_2 = -\sqrt{-5}$$

$$\lambda_3 = \lambda_4 = \sqrt{-20}, \qquad \lambda_5 = \lambda_6 = -\sqrt{-20} \tag{10}$$

and their corresponding eigenvectors are

$$\boldsymbol{v}_1 = \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \boldsymbol{v}_3 = \boldsymbol{v}_5 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \boldsymbol{v}_4 = \boldsymbol{v}_6 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{11}$$

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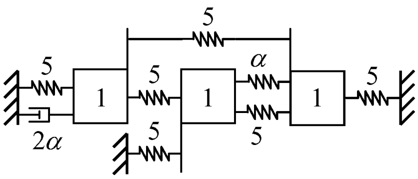

Fig. 1 Three-degree-of-freedom system.

We may apply Eq. (7) to determine the sensitivity of λ_1 and λ_2, yielding $\lambda_1' = \lambda_2' = -\frac{1}{3}$. To confirm this result, we substitute $\alpha = 0.1$ in Eq. (9) and obtain, by direct calculation for the perturbed system,

$$\lambda_1 = \bar{\lambda}_2 = -0.0333467 + 2.2364825i$$

$$\lambda_3 = \bar{\lambda}_5 = -0.0666535 + 4.4703133i$$

$$\lambda_4 = \bar{\lambda}_6 = 4.4944410i \tag{12}$$

Thus, as predicted, λ_1 and λ_2 have been changed by approximately $\alpha\lambda_1' = -1/30$. The change in the other eigenvalues λ_k, $k = 3, 4, 5, 6$, however, cannot be determined by either Eq. (7) or Eq. (8) because they are repeated eigenvalues. Moreover, because any linear combination of $\boldsymbol{v}_3$ and $\boldsymbol{v}_4$ is another eigenvector of the repeated eigenvalues, Eqs. (7) and (8) will generally yield different values for different legitimate choices of eigenvectors.

The objective of this paper is to develop methods for evaluating the sensitivity of the repeated eigenvalues to perturbation.

III. Sensitivity of Regular Repeated Eigenvalues to Perturbations

Consider now the case where λ is a semisimple eigenvalue of multiplicity $p \geq 2$, that is, λ is a p-fold eigenvalue, and there are p linearly independent eigenvectors $\boldsymbol{v}_i$, $i = 1, 2, \ldots, p$, associated with it. It is well known[10–13] that each linear combination of $\boldsymbol{v}_i$ is an eigenvector corresponding to λ. Moreover, in the case of a small perturbation in the system parameters, the eigenvectors associated with the repeated eigenvalue will generally be changed in a discontinuous manner. However, depending on the perturbations, there will be at least one eigenvector that is changed continuously.[11,13] We denote this eigenvector φ. Because φ is formed by a linear combination of $\boldsymbol{v}_i$, we can define the vector $\boldsymbol{r} \in R^P$ via

$$\varphi = \boldsymbol{V}\boldsymbol{r} \tag{13}$$

where $\boldsymbol{V} = [\boldsymbol{v}_1 \;\; \boldsymbol{v}_2 \;\; \cdots \;\; \boldsymbol{v}_p]$. Thus, it follows that

$$(\lambda^2\boldsymbol{M} + \lambda\boldsymbol{C} + \boldsymbol{K})\boldsymbol{V}\boldsymbol{r} = \boldsymbol{0} \tag{14}$$

We multiply Eq. (14) from the right by $\boldsymbol{r}^T\boldsymbol{V}^T$ and define Q as follows:

$$Q = \boldsymbol{r}^T\boldsymbol{V}^T(\lambda^2\boldsymbol{M} + \lambda\boldsymbol{C} + \boldsymbol{K})\boldsymbol{V}\boldsymbol{r} = 0 \tag{15}$$

Differentiating Eq. (15) gives

$$Q' = (\boldsymbol{r}^T\boldsymbol{V}^T)'(\lambda^2\boldsymbol{M} + \lambda\boldsymbol{C} + \boldsymbol{K})\boldsymbol{V}\boldsymbol{r}$$

$$+ \boldsymbol{r}^T\boldsymbol{V}^T(2\lambda\lambda'\boldsymbol{M} + \lambda^2\boldsymbol{M}' + \lambda'\boldsymbol{C} + \lambda\boldsymbol{C}' + \boldsymbol{K}')\boldsymbol{V}\boldsymbol{r}$$

$$+ \boldsymbol{r}^T\boldsymbol{V}^T(\lambda^2\boldsymbol{M} + \lambda\boldsymbol{C} + \boldsymbol{K})(\boldsymbol{V}\boldsymbol{r})' = 0 \tag{16}$$

Because φ is changed continuously, $(\boldsymbol{V}\boldsymbol{r})'$ is bounded, and Eq. (16) is simplified to

$$Q' = \boldsymbol{r}^T\boldsymbol{V}^T(2\lambda\lambda'\boldsymbol{M} + \lambda^2\boldsymbol{M}' + \lambda'\boldsymbol{C} + \lambda\boldsymbol{C}' + \boldsymbol{K}')\boldsymbol{V}\boldsymbol{r} = 0 \tag{17}$$

by virtue of Eq. (14). A sufficient condition for Eq. (17) to be satisfied is that

$$\boldsymbol{V}^T(2\lambda\lambda'\boldsymbol{M} + \lambda^2\boldsymbol{M}' + \lambda'\boldsymbol{C} + \lambda\boldsymbol{C}' + \boldsymbol{K}')\boldsymbol{V}\boldsymbol{r} = \boldsymbol{0} \tag{18}$$

Hence, defining

$$\boldsymbol{A} = \boldsymbol{V}^T(\lambda^2\boldsymbol{M}' + \lambda\boldsymbol{C}' + \boldsymbol{K}')\boldsymbol{V} \tag{19}$$

$$\boldsymbol{B} = -\boldsymbol{V}^T(2\lambda\boldsymbol{M} + \boldsymbol{C})\boldsymbol{V} \tag{20}$$

the derivatives of the repeated eigenvalue λ' are determined by the eigenvalues of

$$(\boldsymbol{A} - \lambda'\boldsymbol{B})\boldsymbol{r} = \boldsymbol{0} \tag{21}$$

This result extends the procedure for finding the derivatives of a repeated eigenvalue obtained by Ojalvo[6] to cover nonconservative vibration systems.

Example 2: Consider again the three-degree-of-freedom system described in example 1 and its modification. To evaluate the derivatives of the repeated eigenvalue $\lambda_3 = \lambda_4 = \sqrt{-20}$, we form the matrix

$$\boldsymbol{V} = [\boldsymbol{v}_3 \quad \boldsymbol{v}_4] = \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ -1 & 1 \end{bmatrix} \tag{22}$$

and obtain, via Eqs. (19) and (20),

$$\boldsymbol{A} = \begin{bmatrix} 1 + 4\sqrt{-5} & -3 + 4\sqrt{-5} \\ -3 + 4\sqrt{-5} & 9 + 4\sqrt{-5} \end{bmatrix} \tag{23}$$

$$\boldsymbol{B} = \begin{bmatrix} -8\sqrt{-5} & 0 \\ 0 & -24\sqrt{-5} \end{bmatrix} \tag{24}$$

We, therefore, have, by Eq. (21),

$$\lambda_3' = -\tfrac{2}{3}, \qquad \lambda_4' = \sqrt{-5}/10 \tag{25}$$

To confirm the result, we use $\hat{\lambda} \cong \lambda + \Delta\alpha\lambda'$, where $\lambda = \sqrt{-20}$ is the eigenvalue before the perturbation; $\Delta\alpha = 0.1$; λ' is the sensitivity from Eq. (25); and $\hat{\lambda}$ is the perturbed eigenvalue. We thus obtain

$$\sqrt{-20} + 0.1\left(-\tfrac{2}{3}\right) = -0.0666667 + 4.4721359i$$

$$\sqrt{-20} + 0.1\sqrt{-5}/10 = 4.4944966i$$

which closely approximate the exact values given in Eq. (12).

IV. Sensitivity of Two Repeated Eigenvalues

For the sake of clarity, we will first address the case of repeated eigenvalues of multiplicity $p = 2$. Denote

$$P(\lambda, \alpha) \hat{=} \det[\lambda^2\boldsymbol{M}(\alpha) + \lambda\boldsymbol{C}(\alpha) + \boldsymbol{K}(\alpha)] = 0 \tag{26}$$

and let $\lambda = \theta$ be two repeated roots of Eq. (26). Then, because θ is a double root of P, we have

$$P(\theta, \alpha) = \left.\frac{\partial P}{\partial \lambda}\right|_{\lambda=\theta} = 0 \tag{27}$$

It follows from Eq. (26) that

$$\mathrm{d}P = \frac{\partial P}{\partial \lambda}\,\mathrm{d}\lambda + \frac{\partial P}{\partial \alpha}\,\mathrm{d}\alpha = 0 \tag{28}$$

and, hence,

$$\left.\frac{\partial P}{\partial \alpha}\right|_{\lambda=\theta} = 0 \tag{29}$$

by virtue of Eq. (27). Then the Taylor series expansion of P, with the third- and higher-order terms in $\Delta\lambda$ and $\Delta\alpha$ neglected, gives

$$P(\lambda + \Delta\lambda, \alpha + \Delta\alpha) = P(\lambda, \alpha) + \Delta\lambda\frac{\partial P}{\partial \lambda} + \Delta\alpha\frac{\partial P}{\partial \alpha}$$

$$+ \left[(\Delta\lambda)^2\frac{\partial^2 P}{\partial \lambda^2}\right]\bigg/2 + \Delta\lambda\Delta\alpha\frac{\partial^2 P}{\partial \lambda\partial \alpha} + \left[(\Delta\alpha)^2\frac{\partial^2 P}{\partial \alpha^2}\right]\bigg/2 = 0 \tag{30}$$

which, in view of Eqs. (26), (27), and (29), yields

$$(\Delta\lambda)^2 \frac{\partial^2 P}{\partial\lambda^2} + 2\Delta\lambda\Delta\alpha \frac{\partial^2 P}{\partial\lambda\partial\alpha} + (\Delta\alpha)^2 \frac{\partial^2 P}{\partial\alpha^2} = 0 \tag{31}$$

Dividing Eq. (31) by $(\Delta\alpha)^2$, we have, in the limit where $\Delta\alpha \to 0$ and $\Delta\lambda \to 0$,

$$(\lambda')^2 \frac{\partial^2 P}{\partial\lambda^2} + 2\lambda' \frac{\partial^2 P}{\partial\lambda\partial\alpha} + \frac{\partial^2 P}{\partial\alpha^2} = 0 \tag{32}$$

The sensitivity of the two repeated eigenvalue are the two roots of Eq. (32).

Remarks: The following criteria are derived from Eq. (32):

1) If $\partial^2 P/\partial\alpha^2 = 0$, then one of the repeated eigenvalues will not be changed by the modification. The rank-one modification discussed by Parlett[14] is a special case of this result.

2) If $\partial^2 P/\partial\lambda^2$ is small compared to $\partial^2 P/\partial\lambda\partial\alpha$, then the sensitivity of one of the repeated eigenvalues is large.

3) If $\partial^2 P/\partial\lambda^2$ and $\partial^2 P/\partial\lambda\partial\alpha$ are small compared to $\partial^2 P/\partial\alpha^2$, then the sensitivity of the two repeated eigenvalues is large.

Example 3: Consider again the three-degree-of-freedom system discussed in examples 1 and 2. The characteristic equation for this system is

$$P = \lambda^6 + 2\lambda^5\alpha + \lambda^4(2\alpha + 45) + 4\lambda^3(\alpha^2 + 15\alpha) + 50\lambda^2(\alpha + 12) + 40\lambda(\alpha^2 + 10\alpha) + 200\alpha + 2000 \tag{33}$$

When $\alpha = 0$, the system has two repeated eigenvalues $\lambda_3 = \lambda_4 = \sqrt{-20}$. To determine the sensitivity of these roots to a small perturbation in α, we evaluate

$$\left.\frac{\partial^2 P}{\partial\lambda^2}\right|_{\lambda=\sqrt{-20},\alpha=0} = 2400 \tag{34}$$

$$\left.\frac{\partial^2 P}{\partial\lambda\partial\alpha}\right|_{\lambda=\sqrt{-20},\alpha=0} = 800 - 60\sqrt{-20} \tag{35}$$

$$\left.\frac{\partial^2 P}{\partial\alpha^2}\right|_{\lambda=\sqrt{-20},\alpha=0} = -80\sqrt{-20} \tag{36}$$

According to Eq. (32), the sensitivity of these eigenvalues are the roots of the quadratic equation

$$2400(\lambda')^2 + \left(1600 - 120\sqrt{-20}\right)\lambda' - 80\sqrt{-20} = 0 \tag{37}$$

that is,

$$\lambda' = -\tfrac{2}{3} \qquad \text{or} \qquad \lambda' = \sqrt{-5}/10 \tag{38}$$

These results are identical to that obtained in Eq. (25) of example 2.

V. Sensitivity of *p* Repeated Eigenvalues

We now consider the case where $\lambda = \theta$ is a repeated root of Eq. (26) of order $p \geq 2$. Here we have

$$P(\theta,\alpha) = \left.\frac{\partial P}{\partial\lambda}\right|_{\lambda=\theta} = \cdots = \left.\frac{\partial^{p-1} P}{\partial\lambda^{p-1}}\right|_{\lambda=\theta} = 0 \tag{39}$$

These conditions, in conjunction with

$$d^k P = 0, \qquad k = 0, 1, \ldots, p-1 \tag{40}$$

imply via the Taylor series expansion of P about λ and α that the sum of pth order terms in the expansion vanishes, that is,

$$\sum_{k=0}^{p} C_k^p \frac{\partial^p P}{\partial^k\lambda\partial^{p-k}\alpha}(\lambda')^k = 0 \tag{41}$$

where C_k^p is the binominal coefficient.

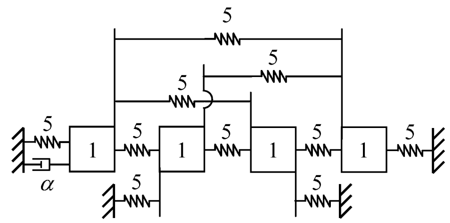

Fig. 2 Four-degree-of-freedom system with three repeated roots when $\alpha = 0$.

Example 4: The mass, spring, and damping matrices of the four-degree-of-freedom system shown in Fig. 2 are

$$\boldsymbol{M} = \boldsymbol{I}_4, \qquad \boldsymbol{K} = \begin{bmatrix} 20 & -5 & -5 & -5 \\ -5 & 20 & -5 & -5 \\ -5 & -5 & 20 & -5 \\ -5 & -5 & -5 & 20 \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{42}$$

When $\alpha = 0$, the system has the following eight eigenvalues:

$$\lambda_1 = \lambda_2 = \lambda_3 = 5i, \qquad \lambda_4 = \lambda_5 = \lambda_6 = -5i$$

$$\lambda_7 = \sqrt{5}i, \qquad \lambda_8 = -\sqrt{5}i \tag{43}$$

that is, this set of eigenvalues consists of two groups of three repeated eigenvalues $\pm 5i$. The characteristic polynomial of the system and its relevant partial derivatives are

$$P(\lambda,\alpha) = \lambda^8 + \alpha\lambda^7 + 80\lambda^6 + 60\alpha\lambda^5 + 2250\lambda^4 + 1125\alpha\lambda^3 + 25{,}000\lambda^2 + 6250\alpha\lambda + 78{,}125 \tag{44}$$

hence,

$$\frac{\partial^3 P}{\partial\lambda^3} = 336\lambda^5 + 210\alpha\lambda^4 + 9600\lambda^3 + 3600\alpha\lambda^2 + 54{,}000\lambda + 6750\alpha \tag{45}$$

$$\frac{\partial^3 P}{\partial\lambda^2\partial\alpha} = 42\lambda^5 + 1200\lambda^3 + 6750\lambda \tag{46}$$

$$\frac{\partial^3 P}{\partial\lambda\partial\alpha^2} = \frac{\partial^3 P}{\partial\alpha^3} = 0 \tag{47}$$

For $p = 3$, Eq. (41) reduces to

$$\frac{\partial^3 P}{\partial\lambda^3}(\lambda')^3 + 3\frac{\partial^3 P}{\partial\lambda^2\partial\alpha}(\lambda')^2 + 3\frac{\partial^3 P}{\partial\lambda\partial\alpha^2}\lambda' + \frac{\partial^3 P}{\partial\alpha^3} = 0 \tag{48}$$

which, on substitution of $\alpha = 0$ and $\lambda = 5i$, gives

$$120{,}000i(\lambda')^3 + 45{,}000i(\lambda')^2 = 0 \tag{49}$$

The roots of Eq. (49) are, thus, $\lambda' = -\frac{3}{8}$ and $\lambda' = 0$ of multiplicity two. To confirm this result, we substitute $\alpha = 0.1$ in Eq. (42) and find by direct calculation that the set of eigenvalues of the perturbed system includes, as expected, two repeated eigenvalues $5i$ and an eigenvalue $-0.0375 + 4.9996i \approx -\frac{3}{8}\alpha + 5i$.

VI. Defective Systems

If a set of repeated eigenvalues of multiplicity p has only $q < p$ linearly independent eigenvectors, then the system is said to be defective. In this case, the sensitivity of the repeated eigenvalues to perturbation cannot, in general, be determined by either the method developed in Sec. III nor by formula (41). The method of Sec. III is

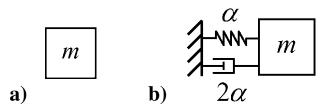

Fig. 3 Defective system a) original system and b) its modification.

not applicable for this case because the hypothesis (13) that φ can be expressed in terms of a linear combination of the eigenvectors does not hold. Equation (41) cannot be used because $d\lambda/d\alpha \to \infty$, and, hence, the partial derivatives of P with respect to α do not generally vanish.

The preceding observation is now demonstrated by the following example.

Example 5: The simplest defective system of all is a single mass m, shown in Fig. 3a. Its differential equation of motion is

$$m\ddot{x} = 0 \tag{50}$$

Suppose that the system is modified by the addition of the spring α and damper 2α, where α is small, as shown in Fig. 3b. The equation of motion for the modified system is

$$m\ddot{x} + 2\alpha\dot{x} + \alpha x = 0 \tag{51}$$

The original system has repeated eigenvalues $\lambda_1 = \lambda_2 = 0$, and the slightly perturbed system has eigenvalues

$$\hat{\lambda}_1 = -\alpha/m - \sqrt{\alpha^2 - m\alpha}\Big/m$$

$$\hat{\lambda}_2 = -\alpha/m + \sqrt{\alpha^2 - m\alpha}\Big/m \tag{52}$$

Hence, by definition,

$$\lambda_1' = \lim_{\alpha\to 0}[(\hat{\lambda}_1 - \lambda_1)/\alpha] = \lim_{\alpha\to 0}\left(-1/m - \sqrt{\alpha^2 - m\alpha}\Big/m\alpha\right)$$

$$\to -1/m - \infty i \tag{53}$$

$$\lambda_2' = \lim_{\alpha\to 0}(\hat{\lambda}_2 - \lambda_2)/\alpha = \bar{\lambda}_1' \tag{54}$$

The application of Eq. (32) together with

$$P = m\lambda^2 + 2\alpha\lambda + \alpha = 0 \tag{55}$$

yields, via

$$2m(\lambda')^2 + 4\lambda' = 0 \tag{56}$$

the erroneous result $\lambda_1' = 0$ and $\lambda_2' = -2/m$.

VII. Conclusions

We have developed two methods for evaluating the derivatives of a repeated eigenvalue of a damped vibratory system with respect to a parameter. The first method generalizes the classical result obtained by Ojalvo[6] to include viscously damped systems. We then furnished an explicit formula (41), from which the sensitivity of a repeated eigenvalue of multiplicity $p \geq 2$ can be found without employing the eigenvector data.

Note that the preceding developments are applicable only to regular systems. A nonsimple eigenvalue, which renders the system defective, is usually not a continuous function of the system's parameters and, hence, generally has no finite derivatives. This issue is demonstrated via example 5 of the paper.

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References

[1]Friedland, S., Nocedal, J., and Overton, M. L., "The Formulation and Analysis of Numerical Methods for Inverse Eigenvalue Problems," *SIAM Journal on Numerical Analysis*, Vol. 24, No. 3, 1987, pp. 634–667.

[2]Joseph, K. T., "Inverse Eigenvalue Problem in Structural Design," *AIAA Journal*, Vol. 30, No. 12, 1992, pp. 2890–2896.

[3]Elhay, S., and Ram, Y. M., "An Affine Inverse Eigenvalue Problem," *Inverse Problems*, Vol. 18, No. 2, 2002, pp. 455–466.

[4]Fox, R. L., and Kapoor, M. P., "Rate of Change of Eigenvalues and Eigenvectors," *AIAA Journal*, Vol. 6, No. 12, 1968, pp. 2426–2429.

[5]Nelson, R. B., "Simplified Calculation of Eigenvectors Derivatives," *AIAA Journal*, Vol. 14, No. 9, 1976, pp. 1201–1205.

[6]Ojalvo, I. U., "Efficient Computation of Modal Sensitivities for Systems with Repeated Frequencies," *AIAA Journal*, Vol. 26, No. 3, 1988, pp. 361–366.

[7]Chen, T. Y., "Design Sensitivity Analysis for Repeated Eigenvalues in Structural Design," *AIAA Journal*, Vol. 31, No. 12, 1993, pp. 2347–2350.

[8]Prells, U., and Friswell, M. I., "Partial Derivatives of Repeated Eigenvalues and Their Eigenvectors," *AIAA Journal*, Vol. 35, No. 8, 1997, pp. 1363–1368.

[9]Prells, U., and Friswell, M. I., "Calculating Derivatives of Repeated and Nonrepeated Eigenvalues Without Explicit Use of Eigenvectors," *AIAA Journal*, Vol. 38, No. 8, 2000, pp. 1426–1436.

[10]Ram, Y. M., Blech, J. J., and Braun, S. G., "Eigenproblem Error Bounds with Application to the Symmetric Dynamic System Modification," *SIAM Journal on Matrix Analysis and Applications*, Vol. 11, No. 4, 1990, pp. 553–564.

[11]Ram, Y. M., and Braun, S. G., "Eigenvector Error Bounds and Their Applications to Structural Modification," *AIAA Journal*, Vol. 31, No. 4, 1993, pp. 759–764.

[12]Crawford, C. R., "A Stable Generalized Eigenvalue Problem," *SIAM Journal on Numerical Analysis*, Vol. 13, No. 6, 1976, pp. 854–860.

[13]Davis, C., and Kahan, W. M., "Rotation of Eigenvectors by a Perturbation, III," *SIAM Journal on Numerical Analysis*, Vol. 7, No. 1, 1970, pp. 1–46.

[14]Parlett, B. N., *The Symmetric Eigenvalue Problem*, Prentice–Hall, Englewood Cliffs, NJ, 1980, pp. 191–194.

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